

Construction of gauge-invariant variables of linear metric perturbations on an arbitrary background spacetime

Kouji Nakamura¹

¹ *TAMA Project, Optical and Infrared Astronomy Division,
National Astronomical Observatory of Japan,
2-21-1, Osawa, Mitaka, Tokyo 181-8588, Japan*

**E-mail: kouji.nakamura@nao.ac.jp*

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An outline of a proof of the decomposition of linear metric perturbations into gauge-invariant and gauge-variant parts on the an arbitrary background spacetime which admits ADM decomposition is discussed. We explicitly construct the gauge-invariant and gauge-variant parts of the linear metric perturbations through the assumption of the existence of some Green functions. We also confirm the result through another approach. This implies that we can develop the higher-order gauge-invariant perturbation theory on an arbitrary background spacetime. Remaining issues to complete the general-framework of the general-relativistic higher-order gauge-invariant perturbation theories are also discussed.
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1. Introduction

Perturbation theories are powerful techniques in many area of physics and the developments of perturbation theories lead physically fruitful results and interpretations of natural phenomena.

In physics, researchers want to describe realistic situations in a compact manner. Exact solutions in a theory for natural phenomena are candidates which can describe realistic situations. However, in many cases, realistic situations are too complicated and often difficult to describe by an exact solution of a theory. This difficulty may be due to the fact that exact solutions only describe special cases even if the theory is appropriate to describe the natural phenomena, or may be due to the lack of the applicability of the theory itself. Even in the case where an exact solution of a theory well describes a physical situation, the properties of the natural system will not be completely described only through the exact solution. In natural phenomena, there always exist “fluctuations”. In this case, perturbative treatments of the theory is a powerful tool and researchers investigate perturbative approach within a theory to clarify the properties of fluctuations.

General relativity is one of theories in which the construction of exact solutions is not so easy. Although there are many exact solutions to the Einstein equation [1], these are often too idealized. Of course, there are some exact solutions to the Einstein equation which well-describe our universe, or gravitational field of stars and black holes. These exact solutions by itself do not describe fluctuations around these exact solutions. To describe them, we have

to consider the perturbations. Therefore, general relativistic *linear* perturbation theory is a useful technique to investigate the properties of fluctuations around exact solutions [2–4]. Through these linear perturbation theories, we can describe fluctuations such as the density or the temperature fluctuations of our universe, gravitational waves from self-gravitating objects.

On the other hand, *higher-order* general-relativistic perturbations also have very wide applications. Among these applications, second-order cosmological perturbations are topical subject [5–9] due to the precise measurements in recent cosmology [10]. Higher-order black hole perturbations are also discussed in some literature [11]. Moreover, as a special example of higher-order perturbation theory, there are researches on perturbations of a spherical star [12], which are motivated by the investigation of the oscillatory behaviors of a rotating neutron star. Thus, there are many physical situations to which general relativistic higher-order perturbation theory should be applied.

As well-known, general relativity is based on the concept of general covariance. Intuitively speaking, the principle of general covariance states that there is no preferred coordinate system in nature, and the notion of general covariance is mathematically included in the definition of a spacetime manifold in a trivial way. This is based on the philosophy that coordinate systems are originally chosen by us, and that natural phenomena have nothing to do with our coordinate system. Due to this general covariance, the “gauge degree of freedom”, which is unphysical degree of freedom of perturbations, arises in general-relativistic perturbations. To obtain physical results, we have to fix this gauge degrees of freedom or to extract some invariant quantities of perturbations. This situation becomes more complicated in higher-order perturbation theory. In some linear perturbation theories on some background spacetimes, there are so-called *gauge-invariant* perturbation theories. In these theories, one may treat only variables which are independent of gauge degree of freedom without any gauge fixing. Therefore, it is worthwhile to investigate higher-order gauge-invariant perturbation theory from a general point of view.

According to these motivations, the general framework of higher-order general-relativistic gauge-invariant perturbation theory has been discussed in some papers [13, 14] by the present author. Although these development of higher-order perturbation theory was originally motivated by researches on the oscillatory behaviors of a self-gravitating Nambu-Goto membrane [15], these works are applicable to cosmological perturbations and we clarified the gauge-invariance of the second-order perturbations of the Einstein equations [9, 16, 17].

In Ref. [13], we proposed a procedure to find gauge-invariant variables for higher-order perturbations on an arbitrary background spacetime. This proposal is based on the single assumption that *we already know the procedure to find gauge-invariant variables for linear-order metric perturbations* (Conjecture 1 in Sec. 2 in this paper). Under the same assumption, we summarize some formulae for the second-order perturbations of the curvatures and energy-momentum tensor for the matter fields in Refs. [14, 16]. Confirming that the above assumption in the case of cosmological perturbations is correct, in Refs. [9], we develop the second-order gauge-invariant cosmological perturbation theory. Through these works, we find that our general framework of higher-order gauge-invariant perturbation theory is well-defined except for the above assumption for linear-order metric perturbations. Therefore, we proposed the above assumption as a conjecture in Ref. [16]. If this conjecture is

true, the higher-order general-relativistic gauge-invariant perturbation theory is completely formulated on an arbitrary background spacetime and has very wide applications.

The main purpose of this paper is to give a scenario of a proof of this conjecture based on the premise that the background spacetime admits ADM decomposition. We explicitly construct the gauge-invariant and gauge-variant parts of the linear metric perturbation. Although some special modes are excluded in the proof in this paper, we may say that the above conjecture is almost correct for linear-order perturbations on an arbitrary background spacetime. This paper is the full paper version of our previous short letter [18].

The organization of this paper is as follows. In Sec. 2, we briefly review the general framework of the second-order gauge-invariant perturbation theory developed in Refs. [13, 14] and the above conjecture is also declared as Conjecture 1 in this review. In Sec. 3, we give a scenario of a proof of Conjecture 1. We note that we assume that the existence of Green functions for two elliptic differential operators in our outline of a proof. Therefore, the modes which belong to the kernel of these two elliptic differential operators are excluded in this paper. Since we use tricky logic in our outline in Sec. 3, we reconsider the derivation of gauge-transformation rules through an alternative approaches in Sec. 4 to check the consistency of our result in Sec. 3. The final section (Sec. 5) is devoted to summary and discussions. Throughout this paper, we use the covariant decomposition of symmetric tensors on Riemannian manifold which was developed by York [26]. We review York’s discussions on this covariant decomposition in Appendix A. Although his discussions are for tensors on a closed manifold, we use his decomposition on a finite region in a Riemannian manifold with boundaries assuming the existence of Green functions for two elliptic differential operators.

We employ the notation in Refs. [13, 14] and use abstract index notation [19]. We also employ natural units in which the velocity of light satisfies $c = 1$.

2. General framework of the higher-order gauge-invariant perturbation theory

In this section, we briefly review the general framework of the gauge-invariant perturbation theory developed in Ref. [13]. The aim of this section is to emphasize that Conjecture 1 is an important premise of our general framework. In Sec. 2.1, we review the notion of the *gauge* in general relativity and *gauge degree of freedom* in general-relativistic perturbations. In Sec. 2.2, the definition of perturbations in general relativity and its gauge-transformation rules are reviewed. When we consider perturbations in any theory with general covariance, we have to exclude these gauge degrees of freedom in the perturbations. To accomplish this, *gauge-invariant variables* of perturbations are useful, and these are regarded as physically meaningful quantities. In Sec. 2.3, a procedure to find gauge-invariant variables of perturbations is explained, which was developed in Ref. [13]. We emphasize that the ingredients of this section do not depend on the details of the background spacetime, except for Conjecture 1.

2.1. Notion of “gauge” in general relativity

General relativity is a theory based on general covariance, which intuitively states that there is no preferred coordinate system in nature. Due to this general covariance, the notion of “gauge” is introduced in the theory. Sachs [20] is the first person who pointed out that there are two kinds of “gauges” in general relativity. He called these two “gauges” as the first- and the second-kind of gauges, respectively. The distinction of these two different notion

of “gauges” is an important premise in the arguments in Sec. 3. Therefore, first of all, we remind the difference of these two “gauges”.

The first kind gauge is a coordinate system on a single manifold \mathcal{M} . On a manifold, we can always introduce a coordinate system as a diffeomorphism ψ_α which maps from an open set $O_\alpha \subset \mathcal{M}$ to $\psi_\alpha(O_\alpha) \subset \mathbb{R}^{n+1}$ (where $n+1 = \dim \mathcal{M}$). This coordinate system ψ_α is called *gauge choice of the first kind*. If we consider another open set $O_\beta \subset \mathcal{M}$, we have another gauge choice $\psi_\beta : O_\beta \mapsto \psi_\beta(O_\beta) \subset \mathbb{R}^{n+1}$. If $O_\alpha \cap O_\beta \neq \emptyset$, we can consider the diffeomorphism $\psi_\beta \circ \psi_\alpha^{-1}$, which is a coordinate transformation : $\psi_\alpha(O_\alpha \cap O_\beta) \subset \mathbb{R}^{n+1} \mapsto \psi_\beta(O_\alpha \cap O_\beta) \subset \mathbb{R}^{n+1}$. This coordinate transformation $\psi_\beta \circ \psi_\alpha^{-1}$ is also called *gauge transformation of the first kind* in general relativity.

On the other hand, *the second kind gauge* appears in perturbation theories in a theory with general covariance. This is the main issue of this paper. In perturbation theories, we always treat two spacetime manifolds. One is the physical spacetime \mathcal{M} which is our nature itself and we want to clarify the properties of \mathcal{M} through perturbations. Another is the background spacetime \mathcal{M}_0 which has nothing to do with our nature but is prepared by hand for perturbative analyses. Let us denote the physical spacetime by (\mathcal{M}, \bar{Q}) and the background spacetime by (\mathcal{M}_0, Q_0) , where \bar{Q} is the collection of tensor fields on \mathcal{M} , and Q_0 is the collection of the background values on \mathcal{M}_0 for the collection \bar{Q} on \mathcal{M} . *The gauge choice of the second kind* is the point identification map $\mathcal{X} : \mathcal{M}_0 \mapsto \mathcal{M}$ [20, 21]. We have to note that the correspondence \mathcal{X} between points on \mathcal{M}_0 and \mathcal{M} is not unique to the perturbation theory with general covariance. General covariance intuitively means that there is no preferred coordinate system in the theory. Due to general covariance, we have no guiding principle to choose the identification map \mathcal{X} . Actually, as a gauge choice of the second kind, we may choose a different point identification map \mathcal{Y} from \mathcal{X} . This implies that there is degree of freedom in the gauge choice of the second kind. This is *the gauge degree of freedom of the second kind* in perturbation theory. In this understanding, *the gauge transformation of the second kind* is a change $\mathcal{X} \rightarrow \mathcal{Y}$ of the identification map $\mathcal{M}_0 \mapsto \mathcal{M}$.

2.2. Perturbations in general relativity

To formulate the above second-kind gauge in more detail, we introduce an infinitesimal parameter λ for perturbations. Further, we consider the $(n+1)+1$ -dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$, where $n+1 = \dim \mathcal{M}$ and $\lambda \in \mathbb{R}$. The background spacetime $\mathcal{M}_0 = \mathcal{N}|_{\lambda=0}$ and the physical spacetime $\mathcal{M} = \mathcal{M}_\lambda = \mathcal{N}|_{\mathbb{R}=\lambda}$ are also submanifolds embedded in the extended manifold \mathcal{N} . Each point on \mathcal{N} is identified by a pair, (p, λ) , where $p \in \mathcal{M}_\lambda$, and each point in the background spacetime \mathcal{M}_0 in \mathcal{N} is identified by $\lambda = 0$. Through this construction, the manifold \mathcal{N} is foliated by $(n+1)$ -dimensional submanifolds \mathcal{M}_λ of each λ , and these are diffeomorphic to the physical spacetime \mathcal{M} and the background spacetime \mathcal{M}_0 . The manifold \mathcal{N} has a natural differentiable structure consisting of the direct product of \mathcal{M} and \mathbb{R} . Further, the perturbed spacetimes \mathcal{M}_λ for each λ must have the same differential structure by this construction.

If a tensor field Q_λ is given on each \mathcal{M}_λ , Q_λ is automatically extended to a tensor field on \mathcal{N} by $Q(p, \lambda) := Q_\lambda(p)$, where $p \in \mathcal{M}_\lambda$. Tensor fields on \mathcal{N} obtained through this construction are necessarily “tangent” to each \mathcal{M}_λ , i.e., their normal component to each \mathcal{M}_λ in \mathcal{N} identically vanishes. To consider the basis of the tangent space of \mathcal{N} , we introduce

the normal form $(d\lambda)_a$ and its dual $(\partial/\partial\lambda)^a$, which are normal to each \mathcal{M}_λ in \mathcal{N} . These satisfy $(d\lambda)_a (\partial/\partial\lambda)^a = 1$. $(d\lambda)_a$ and $(\partial/\partial\lambda)^a$ are normal to any tensor field extended from the tangent space on each \mathcal{M}_λ through the above construction. The set consisting of $(d\lambda)_a$, $(\partial/\partial\lambda)^a$, and the basis of the tangent space on each \mathcal{M}_λ is regarded as the basis of the tangent space of \mathcal{N} .

To define perturbations of an arbitrary tensor field \bar{Q} , we have to compare \bar{Q} on the physical spacetime \mathcal{M}_λ with Q_0 on the background spacetime \mathcal{M}_0 through the introduction of a gauge choice of the second kind. The gauge choice of the second kind is made by assigning a diffeomorphism $\mathcal{X}_\lambda : \mathcal{N} \rightarrow \mathcal{N}$ such that $\mathcal{X}_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_\lambda$. The pull-back \mathcal{X}_λ^* , which is induced by the map \mathcal{X}_λ , maps a tensor field \bar{Q} on \mathcal{M}_λ to a tensor field $\mathcal{X}_\lambda^* \bar{Q}$ on \mathcal{M}_0 . Once the definition of the pull-back of the gauge choice \mathcal{X}_λ is given, the perturbations of a tensor field \bar{Q} under the gauge choice \mathcal{X}_λ are simply defined by the evaluation of the Taylor expansion at \mathcal{M}_0 in \mathcal{N} :

$${}^x Q := \mathcal{X}_\lambda^* \bar{Q}_\lambda|_{\mathcal{M}_0} = Q_0 + \lambda {}^{(1)}_{\mathcal{X}} Q + \frac{1}{2} \lambda^2 {}^{(2)}_{\mathcal{X}} Q + O(\lambda^3), \quad (1)$$

where ${}^{(1)}_{\mathcal{X}} Q$ and ${}^{(2)}_{\mathcal{X}} Q$ are the first- and the second-order perturbations of \bar{Q} , respectively.

We also note that these perturbations completely depend on the gauge choice \mathcal{X}_λ . When we have two different gauge choices \mathcal{X}_λ and \mathcal{Y}_λ , we have two different representations of the perturbative expansion of the pulled-backed variables $\mathcal{X}_\lambda^* \bar{Q}_\lambda|_{\mathcal{M}_0}$ in Eq. (1) and $\mathcal{Y}_\lambda^* \bar{Q}_\lambda|_{\mathcal{M}_0}$:

$${}^y Q := \mathcal{Y}_\lambda^* \bar{Q}_\lambda|_{\mathcal{M}_0} = Q_0 + \lambda {}^{(1)}_{\mathcal{Y}} Q + \frac{1}{2} \lambda^2 {}^{(2)}_{\mathcal{Y}} Q + O(\lambda^3), \quad (2)$$

Although these two representations of the perturbations are different from each other, these should be equivalent because of general covariance. This equivalence is guaranteed by the *gauge-transformation rules* between two different gauge choices. The change of the gauge choice from \mathcal{X}_λ to \mathcal{Y}_λ is represented by the diffeomorphism $\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda$. This diffeomorphism Φ_λ is the map $\Phi_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ for each value of $\lambda \in \mathbb{R}$ and does change the point identification. Therefore, the diffeomorphism Φ_λ is regarded as the gauge transformation $\Phi_\lambda : \mathcal{X}_\lambda \rightarrow \mathcal{Y}_\lambda$. The gauge transformation Φ_λ induces a pull-back from the representation ${}^x Q_\lambda$ of the perturbed tensor field Q in the gauge choice \mathcal{X}_λ to the representation ${}^y Q_\lambda$ in the gauge choice \mathcal{Y}_λ . Actually, the tensor fields ${}^x Q_\lambda$ and ${}^y Q_\lambda$, which are defined on \mathcal{M}_0 , are connected by the linear map Φ_λ^* as ${}^y Q_\lambda = \Phi_\lambda^* {}^x Q_\lambda$. According to generic arguments concerning the Taylor expansion of the pull-back of tensor fields on the same manifold [7, 8], it should be expressed the gauge transformation $\Phi_\lambda^* {}^x Q_\lambda$ in the form

$${}^y Q = \Phi_\lambda^* {}^x Q = {}^x Q + \lambda \mathcal{L}_{\xi_{(1)}} {}^x Q + \frac{\lambda^2}{2} \left\{ \mathcal{L}_{\xi_{(2)}} + \mathcal{L}_{\xi_{(1)}}^2 \right\} {}^x Q + O(\lambda^3), \quad (3)$$

where the vector fields $\xi_{(1)}^a$ and $\xi_{(2)}^a$ are the generators of the gauge transformation Φ_λ . Substituting Eqs. (1) and (2) into Eq. (3), we obtain the gauge-transformation rules for perturbations ${}^{(1)}_{\mathcal{X}} Q$ and ${}^{(2)}_{\mathcal{X}} Q$ as follows:

$${}^{(1)}_{\mathcal{Y}} Q - {}^{(1)}_{\mathcal{X}} Q = \mathcal{L}_{\xi_{(1)}} Q_0, \quad (4)$$

$${}^{(2)}_{\mathcal{Y}} Q - {}^{(2)}_{\mathcal{X}} Q = 2 \mathcal{L}_{\xi_{(1)}} {}^{(1)}_{\mathcal{X}} Q + \left\{ \mathcal{L}_{\xi_{(2)}} + \mathcal{L}_{\xi_{(1)}}^2 \right\} Q_0. \quad (5)$$

The notion of gauge invariance considered in this paper is the *order by order gauge invariance* proposed in Ref. [16]. We call the k th-order perturbation ${}^{(k)}_{\mathcal{X}} Q$ is gauge invariant

iff

$${}^{(k)}_{\mathcal{X}}Q = {}^{(k)}_{\mathcal{Y}}Q \quad (6)$$

for any gauge choice \mathcal{X}_λ and \mathcal{Y}_λ . Through this concept of the order by order gauge invariance, we can decompose any perturbation of Q into the gauge-invariant and gauge-variant parts, as shown in Ref. [13]. In terms of these gauge-invariant variables, we can develop the gauge-invariant perturbation theory. However, this development is based on a non-trivial conjecture, i.e., Conjecture 1 for linear order metric perturbations as explained below.

2.3. Gauge-invariant variables

Inspecting the gauge-transformation rules (4) and (5), we define gauge-invariant variables for metric perturbations and for perturbations of an arbitrary matter field. First, we consider the metric perturbation. The metric \bar{g}_{ab} on \mathcal{M} , which is pulled back to \mathcal{M}_0 using a gauge choice \mathcal{X}_λ , is expanded in the form of Eq. (1):

$$\mathcal{X}_\lambda^* \bar{g}_{ab} = g_{ab} + \lambda \mathcal{X} h_{ab} + \frac{\lambda^2}{2} \mathcal{X}^2 h_{ab} + O^3(\lambda), \quad (7)$$

where g_{ab} is the metric on \mathcal{M}_0 . Of course, the expansion (7) of the metric depends entirely on the gauge choice \mathcal{X}_λ . Nevertheless, henceforth, we do not explicitly express the index of the gauge choice \mathcal{X}_λ if there is no possibility of confusion.

Based on these setups, in Ref. [13], we proposed a procedure to construct gauge-invariant variables for higher-order perturbations. Our starting point to construct gauge-invariant variables is the following conjecture for the linear-order metric perturbation h_{ab} defined by Eq. (7):

Conjecture 1. *If there is a symmetric tensor field h_{ab} of the second rank, whose gauge transformation rule is*

$$\mathcal{Y} h_{ab} - \mathcal{X} h_{ab} = \mathcal{L}_{\xi_{(1)}} g_{ab}, \quad (8)$$

then there exist a tensor field \mathcal{H}_{ab} and a vector field X^a such that h_{ab} is decomposed as

$$h_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab}, \quad (9)$$

where \mathcal{H}_{ab} and X^a are transformed as

$$\mathcal{Y} \mathcal{H}_{ab} - \mathcal{X} \mathcal{H}_{ab} = 0, \quad \mathcal{Y} X^a - \mathcal{X} X^a = \xi_{(1)}^a \quad (10)$$

under the gauge transformation (8), respectively.

In this conjecture, \mathcal{H}_{ab} is gauge-invariant and we call \mathcal{H}_{ab} as *gauge-invariant part* of the perturbation h_{ab} . On the other hand, the vector field X^a in Eq. (9) is gauge dependent, and we call X^a as *gauge-variant part* of the perturbation h_{ab} .

The main purpose of this paper is to give an outline of a proof of Conjecture 1. In the case of cosmological perturbations on a homogeneous and isotropic universe, we confirmed that Conjecture 1 is almost correct except for some special modes of perturbations, and then we could develop the second-order cosmological perturbation theory in a gauge-invariant manner [9]. On the other hand, in the case of the perturbation theory on an arbitrary background spacetime, this conjecture is a highly non-trivial statement due to the non-trivial curvature of the background spacetime, though its inverse statement is trivial. We will see this situation in detail in Sec. 3.

Before going to our outline of a proof of Conjecture 1, we explain how the higher-order gauge-invariant perturbation theory is developed based on this conjecture, here. Through this explanation, we emphasize the importance of Conjecture 1.

As shown in Ref. [13], the second-order metric perturbations l_{ab} are decomposed into gauge-invariant and gauge-variant parts through Conjecture 1. Actually, using the gauge-variant part X^a of the linear-order metric perturbation h_{ab} , we consider the tensor field \hat{L}_{ab} defined by

$$\hat{L}_{ab} := l_{ab} - 2\mathcal{L}_X h_{ab} + \mathcal{L}_X^2 g_{ab}. \quad (11)$$

Through the gauge-transformation rules (5) and (10) for l_{ab} and X^a , respectively, the gauge-transformation rule for this variable \hat{L}_{ab} is given by

$$\mathcal{Y}\hat{L}_{ab} - \mathcal{X}\hat{L}_{ab} = \mathcal{L}_\sigma g_{ab}, \quad \sigma^a := \xi_{(2)}^a + [\xi_{(1)}, X]^a. \quad (12)$$

This is identical to the gauge-transformation rule (8) in Conjecture 1. Therefore, we may apply Conjecture 1 to the variable \hat{L}_{ab} and we can decompose it as

$$\hat{L}_{ab} = \mathcal{L}_{ab} + \mathcal{L}_Y g_{ab}, \quad (13)$$

where the gauge-transformation rules for \mathcal{L}_{ab} and Y^a are given by

$$\mathcal{Y}\mathcal{L}_{ab} - \mathcal{X}\mathcal{L}_{ab} = 0, \quad \mathcal{Y}Y^a - \mathcal{X}Y^a = \xi_{(2)}^a + [\xi_{(1)}, X]^a. \quad (14)$$

Thus, we have accomplished the decomposition of the second-order metric perturbation l_{ab} as

$$l_{ab} =: \mathcal{L}_{ab} + 2\mathcal{L}_X h_{ab} + (\mathcal{L}_Y - \mathcal{L}_X^2) g_{ab}. \quad (15)$$

Furthermore, as shown in Ref. [13], using the first- and second-order gauge-variant parts, X^a and Y^a , of the metric perturbations, the gauge-invariant variables for an arbitrary tensor field Q other than the metric are given by

$$^{(1)}Q := ^{(1)}Q - \mathcal{L}_X Q_0, \quad (16)$$

$$^{(2)}Q := ^{(2)}Q - 2\mathcal{L}_X ^{(1)}Q - \{\mathcal{L}_Y - \mathcal{L}_X^2\} Q_0. \quad (17)$$

It is straightforward to confirm that the variables $^{(1)}Q$ and $^{(2)}Q$ defined by (16) and (17), respectively, are gauge invariant under the gauge-transformation rules (4) and (5), respectively. We have to emphasize that not only gauge-invariant parts of the metric perturbations but also gauge-variant parts X^a and Y^a for metric perturbations play crucial role in these systematic construction of gauge invariant variables $^{(1)}Q$ and $^{(2)}Q$ through Eqs. (16) and (17).

Equations (16) and (17) have an important implication. To see this, we represent these equations as

$$^{(1)}Q = ^{(1)}Q + \mathcal{L}_X Q_0, \quad (18)$$

$$^{(2)}Q = ^{(2)}Q + 2\mathcal{L}_X ^{(1)}Q + \{\mathcal{L}_Y - \mathcal{L}_X^2\} Q_0. \quad (19)$$

These equations imply that any perturbation of first and second order can always be decomposed into their gauge-invariant and gauge-variant parts as Eqs. (18) and (19), respectively. The decomposition formulae (18) and (19) are important consequences in our higher-order

gauge-invariant perturbation theory. Actually, in Ref. [14], we have derived the formulae for the perturbations of the spacetime curvatures and showed that all of these are decomposed as Eqs. (18) and (19). In addition to the spacetime curvatures, in Ref. [16], we also summarized the formulae for the perturbations of the energy-momentum tensors for a single scalar field, a perfect fluid, and an imperfect fluid, and showed that all these energy-momentum tensors and the equations of motion for the matter fields are decomposed into their gauge-invariant and gauge-variant parts as Eqs. (18) and (19). As a result of these decompositions, we can easily show that order by order perturbative equations for any equation on an arbitrary background spacetime are automatically given in gauge-invariant form [14, 16]. Furthermore, we explicitly derived of the second-order Einstein equations for cosmological perturbations in a gauge-invariant manner [9, 16].

We can also expect that the similar structure of equations of the systems will be maintained in any order perturbations and our general framework will be applicable to any order general-relativistic perturbations. Actually, decomposition formulae for the third-order perturbations in two-parameter case, which correspond to Eqs. (18) and (19), are explicitly given in Ref. [13]. Therefore, the similar development is possible for the third-order perturbations. Since we could not find any difficulties to extend higher-order perturbations [13] except for the necessity of long cumbersome calculations, we can construct any order perturbation theory in gauge-invariant manner, recursively.

We note that, through the above recursive procedure, we can find gauge-invariant variables for any perturbative variables without any gauge fixing. The concept of gauge invariance of perturbations should be equivalent to “complete gauge-fixing”. Therefore, we may say that our procedure gives a complete gauge-fixing without any explicit gauge-fixing. We also note that the specification of gauge-variant parts is not unique. The different specifications of gauge-variant variables X^a and Y^a correspond to the different gauge-fixing as explicitly shown in Ref. [24]. In many literature, the explicit gauge-fixing procedures were proposed and these correspond to the explicit specification of the gauge-variant parts X^a and Y^a . However, in this paper, we do not explicitly specify these gauge-variant variables, though we can specify these gauge-variant variables at any time. This is due to the fact that the key idea of our recursive procedure to find gauge-invariant variables is not in the explicit form of the gauge-variant parts but in the gauge-transformation rules of gauge-variant variables of metric perturbations. For example, if we explicitly specify the first-order gauge-variant part X^a , it will be difficult to construct gauge-invariant variables for the second-order metric perturbations, because the gauge-transformation rule of the gauge-variant variable X^a is used to find gauge-invariant variables for the second-order metric perturbations in our recursive procedure.

Finally, we have to emphasize that the above general framework of the higher-order gauge-invariant perturbation theory are independent of the explicit form of the background metric g_{ab} except for Conjecture 1, and are valid not only in cosmological perturbation case but also the other generic situations if Conjecture 1 is true. This implies that if we prove Conjecture 1 for an arbitrary background spacetime in some sense, the above general framework is applicable to perturbation theories on any background spacetime. This is the reason why we proposed Conjecture 1 in Ref. [16]. In the next section, we give a scenario of a proof of Conjecture 1 on an arbitrary background spacetime which admits ADM decomposition.

3. Decomposition of the linear-order metric perturbation

Here, we give a scenario of a proof of Conjecture 1 in Sec. 3.1 through the assumption of the existence of some Green functions of elliptic type derivative operators. This scenario is just an extension of the proof in the case of cosmological perturbations. The comparison with the case of cosmological perturbations in Refs. [9] is discussed in Sec. 3.2.

3.1. A scenario of a proof of Conjecture 1

Now, we give a scenario of a proof of Conjecture 1 on an arbitrary background spacetime. To do this, we assume that the background spacetimes admit ADM decomposition. Therefore, the background spacetime \mathcal{M}_0 (at least the portion of \mathcal{M}_0 which we are addressing) considered here is $n + 1$ -dimensional spacetime which is described by the direct product $\mathbb{R} \times \Sigma$. Here, \mathbb{R} is a time direction and Σ is the spacelike hypersurface ($\dim \Sigma = n$) embedded in \mathcal{M}_0 . This means that \mathcal{M}_0 is foliated by the one-parameter family of spacelike hypersurface $\Sigma(t)$, where $t \in \mathbb{R}$ is a time function. In this setup, the metric on \mathcal{M}_0 is described by

$$g_{ab} = -\alpha^2(dt)_a(dt)_b + q_{ij}(dx^i + \beta^i dt)_a(dx^j + \beta^j dt)_b, \quad (20)$$

where α is the lapse function, β^i is the shift vector, and $q_{ab} = q_{ij}(dx^i)_a(dx^j)_b$ is the metric on $\Sigma(t)$.

Since the ADM decomposition (20) of the metric is a local decomposition, we may regard that the arguments in this paper are restricted to that for a single patch in \mathcal{M}_0 which is covered by the metric (20). Further, we may change the region which is covered by the metric (20) through the choice of the lapse function α and the shift vector β^i . The choice of α and β^i is regarded as the first-kind gauge choice explained in Sec. 2.1, which have nothing to do with the second-kind gauge as emphasized in Sec. 2.1. Since we may regard that the representation (20) of the background metric is that on a single patch in \mathcal{M}_0 , in general situation, each Σ may have its boundary $\partial\Sigma$. For example, in asymptotically flat spacetimes, $\partial\Sigma$ includes asymptotically flat regions [19]. Furthermore, if necessary, we may regard that $\Sigma(t)$ is a portion of the spacelike hypersurface in \mathcal{M}_0 and add disjoint components to the boundary $\partial\Sigma$. For example, when the formation of black holes occurs, we may exclude the region inside the black holes from Σ . In any case, when we consider the spacelike hypersurface Σ with boundary $\partial\Sigma$, we have to impose appropriate boundary conditions at the boundary $\partial\Sigma$.

To consider the decomposition (9) of h_{ab} , we first consider the components of the metric h_{ab} as

$$h_{ab} = h_{tt}(dt)_a(dt)_b + 2h_{ti}(dt)_a(dx^i)_b + h_{ij}(dx^i)_a(dx^j)_b. \quad (21)$$

The components h_{tt} , h_{ti} , and h_{ij} are regarded as a scalar function, components of a vector field, and the components of a symmetric tensor field on the spacelike hypersurface Σ , respectively. Under the gauge-transformation rule (8), the components $\{h_{tt}, h_{ti}, h_{ij}\}$ are

transformed as

$$\begin{aligned} y h_{tt} - x h_{tt} &= 2\partial_t \xi_t - \frac{2}{\alpha} (\partial_t \alpha + \beta^i D_i \alpha - \beta^j \beta^i K_{ij}) \xi_t \\ &\quad - \frac{2}{\alpha} (\beta^i \beta^k \beta^j K_{kj} - \beta^i \partial_t \alpha + \alpha q^{ij} \partial_t \beta_j \\ &\quad + \alpha^2 D^i \alpha - \alpha \beta^k D^i \beta_k - \beta^i \beta^j D_j \alpha) \xi_i, \end{aligned} \quad (22)$$

$$y h_{ti} - x h_{ti} = \partial_t \xi_i + D_i \xi_t - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) \xi_t - \frac{2}{\alpha} M_i{}^j \xi_j, \quad (23)$$

$$y h_{ij} - x h_{ij} = 2D_{(i} \xi_{j)} + \frac{2}{\alpha} K_{ij} \xi_t - \frac{2}{\alpha} \beta^k K_{ij} \xi_k, \quad (24)$$

where $M_i{}^j$ is defined by

$$M_i{}^j := -\alpha^2 K^j{}_i + \beta^j \beta^k K_{ki} - \beta^j D_i \alpha + \alpha D_i \beta^j. \quad (25)$$

Here, K_{ij} is the components of the extrinsic curvature of Σ in \mathcal{M}_0 and D_i is the covariant derivative associate with the metric q_{ij} ($D_i q_{jk} = 0$). The extrinsic curvature K_{ij} and its trace K are related to the time derivative of the metric q_{ij} by

$$K_{ij} = -\frac{1}{2\alpha} \left[\frac{\partial}{\partial t} q_{ij} - D_i \beta_j - D_j \beta_i \right], \quad K := q^{ij} K_{ij}. \quad (26)$$

We also note that the gauge-transformation rules (22)–(24) represent the gauge-transformation of the second kind which have nothing to do with the gauge-degree of freedom of the first kind as explained in Sec. 2.1. We have to emphasize that the main purpose of this paper is to show how to exclude this gauge degree of freedom of the second kind inspecting gauge-transformation rules (22)–(24).

The essence of our strategy for the explicit construction of the gauge-invariant and gauge-variant parts of the linear metric perturbation is already given in our short paper [18]. In Ref. [18], we consider the simple case where $\alpha = 1$ and $\beta^i = 0$. Our strategy is as follows: we first assume that the existence of the variables X_t and X_i whose gauge-transformation rules are given by

$$y X_t - x X_t = \xi_t, \quad (27)$$

$$y X_i - x X_i = \xi_i. \quad (28)$$

This assumption is confirmed through the explicit construction of the gauge-variant part of the linear-order metric perturbation below. Similar technique is given by Pereira et al. [22] in the perturbations on Bianchi type I cosmology.

Inspecting gauge-transformation rules (22)–(24), we define the symmetric tensor field \hat{H}_{ab} whose components are given by

$$\begin{aligned} \hat{H}_{tt} &:= h_{tt} + \frac{2}{\alpha} (\partial_t \alpha + \beta^i D_i \alpha - \beta^j \beta^i K_{ij}) X_t \\ &\quad + \frac{2}{\alpha} (\beta^i \beta^k \beta^j K_{kj} - \beta^i \partial_t \alpha + \alpha q^{ij} \partial_t \beta_j \\ &\quad + \alpha^2 D^i \alpha - \alpha \beta^k D^i \beta_k - \beta^i \beta^j D_j \alpha) X_i, \end{aligned} \quad (29)$$

$$\hat{H}_{ti} := h_{ti} + \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) X_t + \frac{2}{\alpha} M_i{}^j X_j, \quad (30)$$

$$\hat{H}_{ij} := h_{ij} - \frac{2}{\alpha} K_{ij} X_t + \frac{2}{\alpha} \beta^k K_{ij} X_k. \quad (31)$$

The gauge transformation rules (22)–(24) and our assumptions (27) and (28) give the gauge-transformation rules of the components of \hat{H}_{ab} as follows:

$$\mathcal{Y}\hat{H}_{tt} - \mathcal{X}\hat{H}_{tt} = 2\partial_t\xi_t, \quad (32)$$

$$\mathcal{Y}\hat{H}_{ti} - \mathcal{X}\hat{H}_{ti} = \partial_t\xi_i + D_i\xi_t, \quad (33)$$

$$\mathcal{Y}\hat{H}_{ij} - \mathcal{X}\hat{H}_{ij} = 2D_{(i}\xi_{j)}. \quad (34)$$

Since the components \hat{H}_{it} and \hat{H}_{ij} are regarded as components of a vector and a symmetric tensor on $\Sigma(t)$, respectively, we may apply York's decomposition reviewed in Appendix A to \hat{H}_{ti} and \hat{H}_{ij} :

$$\hat{H}_{ti} =: D_i h_{(VL)} + h_{(V)i}, \quad D^i h_{(V)i} = 0, \quad (35)$$

$$\hat{H}_{ij} =: \frac{1}{n} q_{ij} h_{(L)} + h_{(T)ij}, \quad q^{ij} h_{(T)ij} = 0, \quad (36)$$

$$h_{(T)ij} =: (Lh_{(TV)})_{ij} + h_{(TT)ij}, \quad D^i h_{(TT)ij} = 0, \quad (37)$$

where $(Lh_{(TV)})_{ij}$ is defined by [see Eq. (A5) in Appendix A]

$$(Lh_{(TV)})_{ij} := D_i h_{(TV)j} + D_j h_{(TV)i} - \frac{2}{n} q_{ij} D^l h_{(TV)l}. \quad (38)$$

Equations (33) and (34) give the gauge-transformation rules for the variables $h_{(VL)}$, $h_{(V)i}$, $h_{(L)}$, $h_{(T)ij}$, $h_{(TV)i}$, and $h_{(TT)ij}$.

First, we consider the gauge-transformation rule (33) in terms of the decomposition (35):

$$\mathcal{Y}\hat{H}_{ti} - \mathcal{X}\hat{H}_{ti} = D_i (\mathcal{Y}h_{(VL)} - \mathcal{X}h_{(VL)}) + (\mathcal{Y}h_{(V)i} - \mathcal{X}h_{(V)i}) = \partial_t\xi_i + D_i\xi_t. \quad (39)$$

Taking the divergence of this gauge-transformation rule and through the property $D^i h_{(V)i} = 0$, we obtain

$$\Delta (\mathcal{Y}h_{(VL)} - \mathcal{X}h_{(VL)}) = D^i \partial_t\xi_i + \Delta\xi_t. \quad (40)$$

In this paper, we assume that the existence of the Green function Δ^{-1} of the Laplacian $\Delta := D^i D_i$. Then, we easily obtain the gauge-transformation rule for $h_{(VL)}$ as

$$\mathcal{Y}h_{(VL)} - \mathcal{X}h_{(VL)} = \xi_t + \Delta^{-1} D^k \partial_t\xi_k, \quad (41)$$

where we have ignored the modes which belong to the kernel of the derivative operator Δ . Substituting Eq. (41) into Eq. (39) we obtain the gauge-transformation rule for the variable $h_{(V)i}$:

$$\mathcal{Y}h_{(V)i} - \mathcal{X}h_{(V)i} = \partial_t\xi_i - D_i \Delta^{-1} D^k \partial_t\xi_k. \quad (42)$$

The gauge-transformation rules for $h_{(L)}$ and $h_{(T)ij}$ are given from Eq. (34). Since we consider the decomposition (36), the gauge-transformation rule (34) is given by

$$\mathcal{Y}\hat{H}_{ij} - \mathcal{X}\hat{H}_{ij} = \frac{1}{n} q_{ij} (\mathcal{Y}h_{(L)} - \mathcal{X}h_{(L)}) + (\mathcal{Y}h_{(T)ij} - \mathcal{X}h_{(T)ij}) = 2D_{(i}\xi_{j)}. \quad (43)$$

Taking the trace of Eq. (43), we obtain

$$\mathcal{Y}h_{(L)} - \mathcal{X}h_{(L)} = 2D^i \xi_i. \quad (44)$$

The traceless part of Eq. (43) is given by

$$\mathcal{Y}h_{(T)ij} - \mathcal{X}h_{(T)ij} = (L\xi)_{ij}. \quad (45)$$

Note that the variable $h_{(T)ij}$ is also decomposed as Eq. (37) and the gauge-transformation rules for the variable $h_{(T)ij}$ is given by

$$\begin{aligned} \mathcal{Y}h_{(T)ij} - \mathcal{X}h_{(T)ij} &= (L(\mathcal{Y}h_{(TV)} - \mathcal{X}h_{(TV)}))_{ij} + \mathcal{Y}h_{(TT)ij} - \mathcal{X}h_{(TT)ij} \\ &= (L\xi)_{ij}. \end{aligned} \quad (46)$$

Taking the divergence of Eq. (46), we obtain

$$\mathcal{D}^{jl}(\mathcal{Y}h_{(TV)l} - \mathcal{X}h_{(TV)l} - \xi_l) = 0, \quad (47)$$

where the derivative operator \mathcal{D}^{ij} is defined by

$$\mathcal{D}^{ij} := q^{ij}\Delta + \left(1 - \frac{2}{n}\right)D^iD^j + R^{ij}. \quad (48)$$

Here, R^{ij} is the Ricci curvature on Σ . Properties of the derivative operator \mathcal{D}^{ij} are discussed in Appendix A. Here, we assume the existence of the Green function of the derivative operator \mathcal{D}^{ij} and ignore the modes which belong to the kernel of the derivative operator \mathcal{D}^{ij} . Then, we obtain

$$\mathcal{Y}h_{(TV)l} - \mathcal{X}h_{(TV)l} = \xi_l. \quad (49)$$

Substituting Eq. (49) into (46), we obtain

$$\mathcal{Y}h_{(TT)ij} - \mathcal{X}h_{(TT)ij} = 0. \quad (50)$$

In summary, we have obtained the gauge-transformation rules for the variables \hat{H}_{tt} , $h_{(VL)}$, $h_{(V)i}$, $h_{(L)}$, $h_{(TV)i}$, and $h_{(TT)ij}$ as follows:

$$\mathcal{Y}\hat{H}_{tt} - \mathcal{X}\hat{H}_{tt} = 2\partial_t\xi_t, \quad (51)$$

$$\mathcal{Y}h_{(VL)} - \mathcal{X}h_{(VL)} = \xi_t + \Delta^{-1}D^k\partial_t\xi_k, \quad (52)$$

$$\mathcal{Y}h_{(V)i} - \mathcal{X}h_{(V)i} = \partial_t\xi_i - D_i\Delta^{-1}D^k\partial_t\xi_k, \quad (53)$$

$$\mathcal{Y}h_{(L)} - \mathcal{X}h_{(L)} = 2D^i\xi_i, \quad (54)$$

$$\mathcal{Y}h_{(TV)l} - \mathcal{X}h_{(TV)l} = \xi_l, \quad (55)$$

$$\mathcal{Y}h_{(TT)ij} - \mathcal{X}h_{(TT)ij} = 0. \quad (56)$$

Since the gauge transformation rule (55) coincides with the gauge transformation rule (28) for the variable X_i , we may identify the variable X_i with $h_{(TV)i}$:

$$X_i := h_{(TV)i}. \quad (57)$$

Thus, we have confirmed the existence of the variable X_i . Next, we show the existence of the variable X_t whose gauge-transformation rule is given by Eq. (27). Inspecting these gauge transformation rules (52) and (55), we find the definition of X_t as

$$X_t := h_{(VL)} - \Delta^{-1}D^k\partial_th_{(TV)k}. \quad (58)$$

Actually, the gauge transformation rule for X_t defined by Eq. (58) is given by Eq. (27). This is the desired property for the variable X_t . Thus, we have consistently confirmed the existence of the variables X_t and X_i which was assumed in the definitions (29)–(31) of the components of the tensor field \hat{H}_{ab} . This is the most non-trivial part of the ingredients of

this paper. Definitions (57) and (58) also imply that we may start the construction of the gauge-invariant and gauge-variant variables from the decompositions of the components h_{ti} and h_{ij} which are obtained by the substitution of Eqs. (58) and (57) into Eqs. (30) and (31) with the decomposition formulae (35)–(37). This approach is discussed in Sec. 4.

Now, we construct gauge-invariant variables for the linear-order metric perturbation h_{ab} . First, the gauge-transformation rule (56) shows that $h_{(TT)ij}$ is gauge invariant by itself and we define the gauge-invariant transverse-traceless tensor by

$$\chi_{ij} := h_{(TT)ij}. \quad (59)$$

The transverse-traceless property of χ_{ij} is automatically given by the definition of $h_{(TT)ij}$ in Eqs. (36) and (37). Inspecting the gauge-transformation rules (53) and (55), we define a gauge-invariant vector mode ν_i by

$$\nu_i := h_{(V)i} - \partial_t h_{(TV)i} + D_i \Delta^{-1} D^k \partial_t h_{(TV)k}. \quad (60)$$

Actually, we can easily confirm that the variable ν_i is gauge invariant, i.e., $\mathcal{Y}\nu_i - \mathcal{X}\nu_i = 0$. Through the divergenceless property of the variable $h_{(V)i}$, we easily see the property $D^i \nu_i = 0$. The gauge-invariant variables for scalar modes are defined as follows: First, inspecting gauge-transformation rules (27) and (32), we define the scalar variable Φ by

$$-2\Phi := \hat{H}_{tt} - 2\partial_t X_t = \hat{H}_{tt} - 2\partial_t h_{(VL)} + 2\partial_t \Delta^{-1} D^k \partial_t h_{(TV)k}. \quad (61)$$

Inspecting the gauge-transformation rules (54) and (55), we also define another gauge-invariant variable Ψ by

$$-2n\Psi := h_{(L)} - 2D^i X_i = h_{(L)} - 2D^i h_{(TV)i}. \quad (62)$$

We can easily confirm the gauge invariance of the variables Φ and Ψ through gauge-transformation rules (27), (32), (54), and (55). Here, we choose the factor of Ψ in the definition (62) so that we may regard $\Phi = \Psi$ is Newton's gravitational potential in the four-dimensional Newtonian limit.

In terms of these gauge-invariant variables and the variables X_t and X_i , which are defined by Eqs. (58) and (57), respectively, the original components $\{h_{tt}, h_{ti}, h_{ij}\}$ of the metric perturbation h_{ab} is given by

$$\begin{aligned} h_{tt} = & -2\Phi + 2\partial_t X_t - \frac{2}{\alpha} (\partial_t \alpha + \beta^i D_i \alpha - \beta^j \beta^i K_{ij}) X_t \\ & - \frac{2}{\alpha} \left(\beta^i \beta^k \beta^j K_{kj} - \beta^i \partial_t \alpha + \alpha q^{ij} \partial_t \beta_j \right. \\ & \left. + \alpha^2 D^i \alpha - \alpha \beta^k D^i \beta_k - \beta^i \beta^j D_j \alpha \right) X_i, \end{aligned} \quad (63)$$

$$h_{ti} = \nu_i + D_i X_t + \partial_t X_i - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) X_t - \frac{2}{\alpha} M_i^j X_j, \quad (64)$$

$$h_{ij} = -2\Psi q_{ij} + \chi_{ij} + D_i X_j + D_j X_i + \frac{2}{\alpha} K_{ij} X_t - \frac{2}{\alpha} \beta^k K_{ij} X_k. \quad (65)$$

Equations (63)–(65) imply that we may identify the components of the gauge-invariant variables \mathcal{H}_{ab} and the gauge-variant variable X_a so that

$$\mathcal{H}_{tt} := -2\Phi, \quad \mathcal{H}_{ti} := \nu_i, \quad \mathcal{H}_{ij} := -2\Psi q_{ij} + \chi_{ij} \quad (66)$$

and

$$X_a := X_t(dt)_a + X_i(dx^i)_a. \quad (67)$$

These identifications lead to the decomposition formula (9) for the linear-order metric perturbation on an arbitrary background spacetime.

We note that, in this outline of a proof, we assumed the existence of two Green function of the derivative operators $\Delta := D^i D_i$ and \mathcal{D}^{ij} which is defined by Eq. (48). In other words, we have ignored the modes which belong to the kernel of these derivative operators Δ and \mathcal{D}^{ij} . We call these modes as *zero modes*. To explicitly specify the Green functions for Δ and \mathcal{D}^{ij} , we have to impose boundary conditions at boundaries $\partial\Sigma$. Since the operators Δ and \mathcal{D}^{ij} are elliptic, the change of the boundary conditions at $\partial\Sigma$ is adjusted by functions which belong to the kernel of the operators Δ and \mathcal{D}^{ij} , i.e., zero modes. Therefore, we may say that the information for the boundary conditions for the Green functions Δ^{-1} and $(\mathcal{D}^{ij})^{-1}$ is also included in these zero modes. To take these modes into account, the different treatments will be necessary. We call this issue as *zero-mode problem*.

3.2. Comparison with the FLRW background case

Here, we consider the comparison with the case where the background spacetime \mathcal{M}_0 is a homogeneous isotropic universe discussed in Refs. [9] to clarify differences of the above arguments in Sec. 3.1 from the well-known formulation of cosmological perturbations. The case of a homogeneous isotropic background universe corresponds to the case $\alpha = 1$, $\beta^i = 0$, and $K_{ij} = -Hq_{ij}$, where $H = \partial_t a/a$ and a is the scale factor of the universe.

On this background spacetime, in Refs. [9], we decomposed the components h_{ti} and h_{ij} of the metric perturbation h_{ab} as

$$h_{ti} = \tilde{D}_i \tilde{h}_{(VL)} + \tilde{h}_{(V)i}, \quad \tilde{D}^i \tilde{h}_{(V)i} = 0, \quad (68)$$

$$h_{ij} = a^2 \tilde{h}_{(L)} \gamma_{ij} + a^2 \tilde{h}_{(T)ij}, \quad \gamma^{ij} \tilde{h}_{(T)ij} = 0, \quad (69)$$

$$\tilde{h}_{(T)ij} = \left(\tilde{D}_i \tilde{D}_j - \frac{1}{n} \gamma_{ij} \tilde{\Delta} \right) \tilde{h}_{(TL)} + 2 \tilde{D}_{(i} \tilde{h}_{(TV)j)} + \tilde{h}_{(TT)ij}, \quad (70)$$

$$\tilde{D}^i \tilde{h}_{(TV)i} = 0, \quad \tilde{D}^i \tilde{h}_{(TT)ij} = 0, \quad (71)$$

where $q_{ij} = a^2 \gamma_{ij}$, γ_{ij} is the metric on a maximally symmetric space, \tilde{D}_i is the covariant derivative associated with the metric γ_{ij} , and $\tilde{\Delta} := \tilde{D}^i \tilde{D}_i$. In the case where $\alpha = 1$, $\beta^i = 0$, and $K_{ij} = -Hq_{ij}$, the decomposition (68)–(71) are equivalent to the decomposition (35)–(37) with Eqs. (29)–(31), (57), and (58) in this paper. Therefore, one might think that we may also apply the decomposition (68)–(71) even in the case of an arbitrary background spacetime. However, in the case of an arbitrary background spacetime, the decomposition (68)–(71) is ill-defined. Actually, if we regard that the decomposition (68)–(71) is that for an arbitrary background spacetime, we cannot separate $\tilde{h}_{(TL)}$ and $\tilde{h}_{(TV)j}$ due to the non-trivial curvature terms of the background \mathcal{M}_0 as pointed out by Deser [23]. These curvature terms come from the commutation relation between the covariant derivative D_i and the derivative operator \mathcal{D}^{ij} . This is why we apply the decomposition (35)–(37) with Eqs. (29)–(31), (57), and (58) instead of (68)–(71).

Furthermore, in Refs. [9], we have assumed the existence of Green functions of the derivative operators $\tilde{\Delta}$, $\tilde{\Delta} + (n-1)k$, and $\tilde{\Delta} + nk$ to guarantee the one to one correspondence of

the set $\{h_{tt}, h_{ti}, h_{ij}\}$ and $\{\{h_{tt}, h_{(VL)}, h_{(L)}, h_{(TL)}\}, \{h_{(V)i}, h_{(TV)i}\}, h_{(TT)ij}\}$, where k is the curvature constant on the maximally symmetric space. The special modes which belong to the kernel of the derivative operators $\tilde{\Delta}$, $\tilde{\Delta} + (n-1)k$, and $\tilde{\Delta} + nk$ were not included in the consideration in Refs. [9]. On the other hand, in this paper, we ignore the modes which belong to the kernel of the derivative operator Δ and \mathcal{D}^{ij} . We note that the modes ignored in this paper coincides with the modes ignored in Ref. [9]. Trivially, the above operator $\tilde{\Delta} := \tilde{D}^i \tilde{D}_i$ corresponds to the Laplacian Δ in this paper. In the case of the maximally symmetric n -space, the Riemann curvature and Ricci curvature are given by

$$R_{ijkl} = 2kq_{k[i}q_{j]l} = 2kq_{k[i}q_{j]l}, \quad R_{ik} = q^{jl}R_{ijkl} = (n-1)kq_{ik}. \quad (72)$$

In this case, the derivative operator \mathcal{D}^{ij} defined by Eq. (48) is given by

$$\mathcal{D}^{ij} = q^{ij}(\Delta + (n-1)k) + \left(1 - \frac{2}{n}\right) D^i D^j. \quad (73)$$

When the operator \mathcal{D}^{ij} acts on an arbitrary transverse vector field v_i ($D^i v_i = 0$), we easily see that

$$\mathcal{D}^{ij} v_j = (\Delta + (n-1)k) v^i. \quad (74)$$

Therefore, the kernel of the derivative operator $\tilde{\Delta} + (n-1)k$ in Refs. [9] is included in the kernel of the derivative operator \mathcal{D}^{ij} in this paper. Finally, we consider the case where the derivative operator \mathcal{D}^{jl} acts on the gradient $D_l f$ of an arbitrary scalar function f :

$$\mathcal{D}^{jl} D_l f = 2\frac{n-1}{n} \left[D^j \Delta + \frac{n}{n-1} R^{jl} D_l \right] f. \quad (75)$$

In the case of maximally symmetric n -space, curvature tensors are given by Eqs. (72) and the derivative operator $\mathcal{D}^{jl} D_l$ is given by

$$\mathcal{D}^{jl} D_l f = 2\frac{n-1}{n} D^j (\Delta + nk) f. \quad (76)$$

To solve the equation $\mathcal{D}^{jl} D_l f = g^j$, we have to use the Green functions associated with the derivative operators Δ and $\Delta + nk$. These are the reason for the fact that the Green functions Δ^{-1} , $(\Delta + (n-1)k)^{-1}$, and $(\Delta + nk)^{-1}$ were necessary to guarantee the one-to-one correspondence between the components $\{h_{ti}, h_{ij}\}$ and $\{\tilde{h}_{(VL)}, \tilde{h}_{(V)i}, \tilde{h}_{(L)}, \tilde{h}_{(TL)}, \tilde{h}_{(TV)i}, \tilde{h}_{(TT)ij}\}$ in Eqs. (68)–(71). In other words, we may say that the special modes belong to the kernel of the derivative operators Δ and \mathcal{D}^{ij} are equivalent to the special modes which belong to the kernel of the derivative operators Δ , $\Delta + (n-1)k$, and $\Delta + nk$ in the case of the maximally symmetric space Σ in Refs. [9].

Finally, we note that the gauge-invariant variables defined by Eqs. (59)–(62) are generalizations of the metric perturbation in the longitudinal gauge in cosmological perturbation theory [24]. As noted in Refs. [8, 16, 24], the choice of the gauge-invariant variable is not unique and there are many choice of the gauge-invariant variables. This situation corresponds to the fact that there are infinitely many complete gauge-fixing procedures. Actually, in Ref. [24], through the specification of the gauge-variant parts X^a and Y^a of the first- and the second-order metric perturbations in Eqs. (9) and (15), we realized the two different complete gauge-fixing in the first- and the second-order perturbations, namely, the Poisson gauge and the flat gauge, at kinematical level. Through the similar technique, Ugla and Wainwright [25] derived the linear-order Einstein equations both in the longitudinal gauge (Poisson gauge) and in the flat gauge (constant curvature gauge) in a compact manner.

4. Alternative construction of gauge-variant and gauge-invariant parts

The result obtained in Sec. 3.1 implies that we may define the variables $h_{(VL)}$, $h_{(V)i}$, $h_{(L)}$, $h_{(TV)i}$, $h_{(TT)ij}$ by the following decomposition formulae for the components h_{ti} and h_{ij} :

$$h_{ti} =: D_i h_{(VL)} + h_{(V)i} - \frac{2}{\alpha} \left(D_i \alpha - \beta^k K_{ik} \right) \left(h_{(VL)} - \Delta^{-1} D^k \partial_t h_{(TV)k} \right) - \frac{2}{\alpha} M_i^k h_{(TV)k}, \quad (77)$$

$$h_{ij} =: \frac{1}{n} q_{ij} h_{(L)} + (L h_{(TV)})_{ij} + h_{(TT)ij} + \frac{2}{\alpha} K_{ij} \left(h_{(VL)} - \Delta^{-1} D^k \partial_t h_{(TV)k} \right) - \frac{2}{\alpha} K_{ij} \beta^k h_{(TV)k}, \quad (78)$$

$$D^i h_{(V)i} = 0, \quad q^{ij} h_{(TT)ij} = 0 = D^i h_{(TT)ij}, \quad (79)$$

where M_i^j is defined by Eq. (25). Here, these expressions are obtained through the substitution of Eqs. (57) and (58) into (30) and (31) and York's decomposition (35)–(37). In this section, we check the consistency of our result in Sec. 3.1. To do this, we change the starting point of our arguments to the decomposition formulae (77)–(79), though the starting point of our outline of a proof in Sec. 3.1 was Eqs. (29)–(31) with the assumption of the existence of the variables X_t and X_i . First, we consider the derivation of the inverse relation of Eqs. (77)–(79) in Sec. 4.1. Then, we derive the gauge transformation rules for the variables $h_{(VL)}$, $h_{(V)i}$, $h_{(L)}$, $h_{(TV)i}$, and $h_{(TT)ij}$ in Sec. 4.2.

4.1. Inverse relation

To derive the inverse relation of Eqs. (77)–(79), we first consider Eq. (77). Assuming the existence of the Green function \mathcal{F}^{-1} for the elliptic derivative operator

$$\mathcal{F} := \Delta - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) D^i - 2D^i \left\{ \frac{1}{\alpha} (D_i \alpha - \beta^j K_{ij}) \right\}, \quad (80)$$

we obtain the relations

$$h_{(VL)} = \mathcal{F}^{-1} \left[D^k h_{tk} - D^k \partial_t h_{(TV)k} + D^k \left(\frac{2}{\alpha} M_k^l h_{(TV)l} \right) \right] + \Delta^{-1} D^k \partial_t h_{(TV)k}, \quad (81)$$

$$h_{(V)i} = h_{ti} - D_i \Delta^{-1} D^k \partial_t h_{(TV)k} + \frac{2}{\alpha} M_i^k h_{(TV)k} + \left[D_i - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) \right] \mathcal{F}^{-1} \left[-D^k h_{tk} + D^k \partial_t h_{(TV)k} - D^k \left(\frac{2}{\alpha} M_k^l h_{(TV)l} \right) \right]. \quad (82)$$

Equations (81) and (82) imply that we can obtain the relations between $\{h_{(VL)}, h_{(V)i}\}$ and $\{h_{ti}, h_{ij}\}$ if the relation between $h_{(TV)i}$ and $\{h_{ti}, h_{ij}\}$ is specified. On the other hand, the

trace part and the traceless part of Eq. (78) are given by

$$h_{(L)} = q^{ij}h_{ij} + \frac{2}{\alpha}K\beta^k h_{(TV)k} - \frac{2}{\alpha}K \left(\mathcal{F}^{-1} \left[D^k h_{tk} - D^k \partial_t h_{(TV)k} + D^k \left(\frac{2}{\alpha} M_k^l h_{(TV)l} \right) \right] \right), \quad (83)$$

$$h_{ij} - \frac{1}{n}q_{ij}q^{kl}h_{kl} = (Lh_{(TV)})_{ij} + h_{(TT)ij} - \frac{2}{\alpha}\tilde{K}_{ij}\beta^k h_{(TV)k} + \frac{2}{\alpha}\tilde{K}_{ij}\mathcal{F}^{-1} \left[D^k h_{tk} - D^k \partial_t h_{(TV)k} + D^k \left(\frac{2}{\alpha} M_k^l h_{(TV)l} \right) \right], \quad (84)$$

where we have used Eq. (81) and defined the traceless part \tilde{K}_{ij} of the extrinsic curvature K_{ij} by

$$\tilde{K}_{ij} := K_{ij} - \frac{1}{n}q_{ij}K. \quad (85)$$

Taking the divergence of Eq. (84), we obtain the single integro-differential equation for $h_{(TV)k}$:

$$\begin{aligned} & \mathcal{D}_j^k h_{(TV)k} + D^m \left[\frac{2}{\alpha}\tilde{K}_{mj} \left\{ \mathcal{F}^{-1} D^k \left(\frac{2}{\alpha} M_k^l h_{(TV)l} - \partial_t h_{(TV)k} \right) - \beta^k h_{(TV)k} \right\} \right] \\ &= D^m \left[h_{mj} - \frac{1}{n}q_{mj}q^{kl}h_{kl} - \frac{2}{\alpha}\tilde{K}_{mj}\mathcal{F}^{-1} D^k h_{tk} \right]. \end{aligned} \quad (86)$$

The existence and the uniqueness of the solution to this integro-differential equation is highly non-trivial. However, we assume the existence and the uniqueness of the solution $h_{(TV)k} = h_{(TV)k}[h_{tm}, h_{mn}]$ to this integro-differential equation (86), here. This solution describes the expression of the variable $h_{(TV)i}$ in terms of the original components h_{ti} and h_{ij} of the metric perturbation h_{ab} . Substituting the solution $h_{(TV)k} = h_{(TV)k}[h_{tm}, h_{mn}]$ to Eq. (86) into Eqs. (81)–(83), we can obtain the representation of the variables $h_{(VL)}$, $h_{(V)i}$, $h_{(L)}$ in terms of the original components h_{ti} and h_{ij} of h_{ab} . Furthermore, the representation of the variable $h_{(TT)ij}$ in terms of h_{ti} and h_{ij} are derived from Eq. (84) through the substitution of the solution $h_{(TV)k} = h_{(TV)k}[h_{tm}, h_{mn}]$ to Eq. (86).

Thus, the decomposition formulae (77)–(79) are invertible if the Green functions Δ^{-1} , \mathcal{F}^{-1} exist and the solution to the integro-differential equation (86) exists and is unique.

4.2. Gauge-transformation rules

Through similar calculations to those in Sec. 4.1, we can derive the gauge-transformation rules for the variables $h_{(VL)}$, $h_{(V)i}$, $h_{(L)}$, $h_{(TV)i}$, and $h_{(TT)ij}$. From Eqs. (81) and (82) the gauge-transformation rules (23) for the component h_{it} , we obtain the gauge-transformation

rule for the variable $h_{(VL)}$ and $h_{(V)i}$:

$$\begin{aligned} \mathcal{Y}h_{(VL)} - \mathcal{X}h_{(VL)} &= \xi_t + \Delta^{-1}D^k\partial_t\xi_k \\ &\quad + \mathcal{F}^{-1}D^k\left[-\partial_tA_k + \frac{2}{\alpha}M_k{}^l A_l\right] + \Delta^{-1}D^k\partial_tA_k, \end{aligned} \quad (87)$$

$$\begin{aligned} \mathcal{Y}h_{(V)i} - \mathcal{X}h_{(V)i} &= \partial_t\xi_i - D_i\Delta^{-1}D^k\partial_t\xi_k \\ &\quad + \left[D_i - \frac{2}{\alpha}(D_i\alpha - \beta^j K_{ij})\right]\mathcal{F}^{-1}D^k\left[\partial_tA_k - \frac{2}{\alpha}M_k{}^l A_l\right] \\ &\quad - D_i\Delta^{-1}D^k\partial_tA_k + \frac{2}{\alpha}M_i{}^k A_k, \end{aligned} \quad (88)$$

where $A_i := \mathcal{Y}h_{(TV)i} - \mathcal{X}h_{(TV)i} - \xi_i$. As in the case of the relations (81) and (82), these gauge-transformation rules (87) and (88) imply that we can obtain the gauge-transformation rules for the variable $h_{(VL)}$ and $h_{(V)i}$ if the gauge-transformation rule for the variable $h_{(TV)i}$ is specified.

From Eq. (83) and the gauge-transformation rule (24), we can derive the gauge-transformation rule for the variable $h_{(L)}$:

$$\begin{aligned} \mathcal{Y}h_{(L)} - \mathcal{X}h_{(L)} &= 2D^l\xi_l + \frac{2}{\alpha}K\beta^k A_k \\ &\quad + \frac{2}{\alpha}K\left(\mathcal{F}^{-1}D^k\left[\partial_tA_k - \frac{2}{\alpha}M_k{}^l A_l\right]\right). \end{aligned} \quad (89)$$

As in the case of the gauge-transformation rules (87) and (88), the gauge-transformation rule (89) also implies that we can obtain the gauge-transformation rule for the variable $h_{(L)}$ if the gauge-transformation rule for the variable $h_{(TV)i}$ is specified. On the other hand, from the gauge-transformation rule for the traceless part (84) of h_{ij} , we obtain the equation

$$\begin{aligned} (LA)_{ij} + \mathcal{Y}h_{(TT)ij} - \mathcal{X}h_{(TT)ij} - \frac{2}{\alpha}\tilde{K}_{ij}\beta^k A_k \\ - \frac{2}{\alpha}\tilde{K}_{ij}\mathcal{F}^{-1}D^k\left[\partial_tA_k - \frac{2}{\alpha}M_k{}^l A_l\right] = 0, \end{aligned} \quad (90)$$

where we have used Eqs. (23) and (24). The divergence of Eq. (90) yields

$$\mathcal{D}_j{}^l A_l + D^l\left[\frac{2}{\alpha}\tilde{K}_{lj}\left\{\mathcal{F}^{-1}D^k\left(\partial_tA_k - \frac{2}{\alpha}M_k{}^l A_k\right) - \beta^k A_k\right\}\right] = 0. \quad (91)$$

Here, we note that we have assumed the existence and the uniqueness of the solution to Eq. (86). Since Eq. (91) is the homogeneous version of Eq. (86), this assumption yields that we have the unique solution $A_k = 0$ to Eq. (91), i.e.,

$$\mathcal{Y}h_{(TV)i} - \mathcal{X}h_{(TV)i} = \xi_i. \quad (92)$$

Thus, we have specified the gauge-transformation rule for the variable $h_{(TV)i}$ and the gauge-transformation rule (92) coincides with Eq. (49).

Substituting Eq. (92) into Eqs. (87)–(90), we obtain the gauge-transformation rules for the variables $h_{(VL)}$, $h_{(V)i}$, $h_{(L)}$, and $h_{(TT)ij}$. We easily see that the resulting gauge-transformation rules for these variables are given by Eqs. (41), (42), (44), and (50), respectively. Further, we can construct gauge-variant and gauge-invariant parts of the metric perturbation h_{ab} in the same manner as in Sec. 3.1 and we can confirm Conjecture 1. Thus, we have reached to the same conclusions as those obtained in Sec. 3.1. Therefore, we may say that the results obtained in Sec. 3.1 are consistent.

5. Summary and discussions

In summary, we proposed a scenario of a proof of Conjecture 1 for an arbitrary background spacetime which admits ADM decomposition. Conjecture 1 states that we already know the procedure to decompose the linear-order metric perturbation h_{ab} into its gauge-invariant part \mathcal{H}_{ab} and gauge-variant part X_a . In the cosmological perturbation case, this conjecture was confirmed and then the second-order cosmological perturbation theory was developed in our series of papers [9, 16, 17]. However, as reviewed in Sec. 2, Conjecture 1 is the only non-trivial part when we consider the general framework of gauge-invariant perturbation theory on an arbitrary background spacetime. Although there will be many approaches to prove Conjecture 1, in this paper, we just proposed an outline a proof for an arbitrary background spacetime.

In the outline shown in Sec. 3.1, we assumed the existence of Green functions of the elliptic derivative operators Δ and \mathcal{D}^{ij} . This assumption implies that we have ignored the modes which belong to the kernel of these derivative operators. We call these modes as *zero modes*. To derive the explicit representation of these Green functions, we have to impose appropriate boundary conditions for the perturbative metric at the boundary $\partial\Sigma$. The modes which belong to the kernel of the above derivative operators also includes the degree of freedom of these boundary conditions, because of the ellipticity of the derivative operators Δ and \mathcal{D}^{ij} as noted in Sec. 3.1. For this reason, we also emphasized the importance of these zero modes from a view point of the globalization of gauge-invariant variables [30]. Within the arguments in this paper, there is no information for the treatment of these mode. To discuss these modes, different treatments of perturbations will be necessary. We call this issue as *zero-mode problem*. The situation is similar to the cosmological perturbation case as noted in Sec. 3.2 and zero-mode problem exists even in the cosmological perturbation case.

This zero-mode problem in cosmological perturbations also corresponds to the $l = 0$ and $l = 1$ (even) mode problem in perturbation theories on spherically symmetric background spacetimes. In the perturbation theory on spherically symmetric background spacetimes, one usually considers $2 + 2$ formulation [3] (or $2 + n$ formulation [4]), in which the similar decomposition to Eqs. (69)–(71) is applied. In this case, the indices i, j, \dots in these equations correspond to the indices of the components of a tensor field on S^2 . Since S^2 is a 2-dimensional maximally symmetric space with the positive curvature, we may regard $n = 2$ and $k = 1$ in Sec. 3.2. Then, the three derivative operators Δ , $\Delta + (n - 1)k$, and $\Delta + nk$ in Sec. 3.2 are given by Δ , $\Delta + 1$, and $\Delta + 2$, respectively. Since the eigenvalue of the Laplacian Δ on S^2 is given by $\Delta = -l(l + 1)$, we may say that the modes with $l = 0$ and $l = 1$ belong to the kernel of the derivative operator Δ and $\Delta + nk$, respectively. Therefore, we may say that the problem concerning about the modes with $l = 0$ and $l = 1$ in the perturbations on spherically symmetric background spacetimes is the same problem as the zero-mode problem mentioned above.

Thus, the arguments in this paper show that zero-mode problem generally appears in many perturbation theories in general relativity and we have seen that the appearance of this zero-mode problem from general point of view. Furthermore, as discussed in Appendix A, the kernel of the derivative operator \mathcal{D}^{ij} includes conformal Killing (or Killing) vectors on Σ . Therefore, the existence of zero mode is also related to the symmetry of the background spacetime. To resolve this zero-mode problem, careful discussions on domains of functions for

perturbations and its boundary conditions at $\partial\Sigma$ will be necessary. We leave this zero-mode problem as a future work.

In our outline of a proof in Sec. 3.1, we used a tricky logic to construct gauge-invariant and gauge-variant parts of the metric perturbation h_{ab} . Therefore, in Sec. 4, we confirmed the consistency of our result in Sec. 3.1 through the change of the starting point of our arguments, and then, we have reached to the same conclusion as in Sec. 3.1. Due to this fact, we may say that the result obtained in Sec. 3.1 is consistent. In the approach in Sec. 4, we assume the existence of Green functions Δ^{-1} and \mathcal{F}^{-1} . Further, we also assume the existence and the uniqueness of the solution to the integro-differential equation (86). Although the correspondence between the existence of the Green functions which are assumed in Sec. 3.1 and these assumptions is not clear, we may say that the above zero-mode problem is delicate in the case of an arbitrary background spacetime.

We also emphasize that the gauge issue discussed in this paper is just a kinematical one and we did not use any information of the field equations such as Einstein equations. In other words, the results in this paper is also applicable not only to general relativity but also to any other metric theories of gravity with general covariance.

Since our motivation of this paper is in higher-order general-relativistic perturbation theory, readers might think that the ingredients of this paper is related to the issue so-called “linearization instability” [27]. However, at least within the level of the ingredients of this paper, our arguments themselves still have nothing to do with the issue of linearization instability. The linearization instability is the issue of the existence of the solutions to the initial value constraints in the Einstein equations, while our arguments in this paper is just a kinematical one and have nothing to do with field equations as mentioned above. Although it will be interesting to reconsider the issue of linearization instability from the view point of our gauge-invariant perturbation theory, this issue is beyond the current scope of this paper. To discuss the linearization instability from our view point, it will be necessary to discuss the above zero-mode problem, at first.

Although we should take care of the zero-mode problem, we have almost completed the general framework of the general-relativistic higher-order gauge-invariant perturbation theory. The outline of a proof of Conjecture 1 shown in this paper gives rise to the possibility of the application of our general framework for the higher-order gauge-invariant perturbation theory not only to cosmological perturbations [9, 16, 17] but also to perturbations of black hole spacetimes or perturbations of general relativistic stars through an unified formulation of the general-relativistic perturbation theory. Furthermore, as mentioned above, the results in this paper is also applicable not only to general relativity but also to any other metric theories of gravity with general covariance. Therefore, we may say that the wide applications of our gauge-invariant perturbation theory will be opened due to the discussions in this paper. We also leave these development of gauge-invariant perturbation theories for these background spacetimes as future works.

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A. Covariant orthogonal decomposition of symmetric tensors

Since the each order metric perturbation is regarded as a symmetric tensor on the background spacetime (\mathcal{M}_0, g_{ab}) through an appropriate gauge choice, the covariant decomposition of symmetric tensors is useful and actually used in the main text. Here, we review of the covariant decomposition of symmetric tensors of the second rank on an curved Riemannian manifold based on the work by York [26].

On an arbitrary curved Riemannian space (Σ, q_{ab}) ($\dim \Sigma = n$), one can decompose an arbitrary vector or one-form into its transverse and longitudinal parts as

$$A_a = A_i(dx^i)_a = (D_i A_{(L)} + A_{(V)i})(dx^i)_a, \quad D^i A_{(V)i} = 0, \quad (\text{A1})$$

where D_i is the covariant derivative associated with the metric $q_{ab} = q_{ij}(dx^i)_a(dx^j)_b$. $A_{(L)}$ is called the longitudinal part or the scalar part and $A_{(V)i}$ is called the transverse part or vector part of the vector field A_a on (Σ, q_{ab}) , respectively.

Moreover, this decomposition is not only covariant with respect to arbitrary coordinate transformations, it is also orthogonal in the natural global scalar product. To clarify this orthogonality, York [26] introduced the inner product for the vector fields on Σ . This is, for any two vectors V^a and W^a , we have

$$(V, W) := \int_{\Sigma} \epsilon_q V^a W^b q_{ab}, \quad (\text{A2})$$

where ϵ_q denotes the volume element which makes the integral invariant and the integration extends over the entire manifold (Σ, q_{ab}) . In terms of this inner product, the orthogonality of the vector fields $V^a = D^a V_{(L)} := q^{ab} D_b V_{(L)}$ and $W^a = q^{ab} V_{(V)b}$ with $D^a V_{(V)a} = 0$ is given by

$$\int_{\Sigma} \epsilon_q D_a V_{(L)} V_{(V)b} q^{ab} = \int_{\partial\Sigma} s_a V_{(L)} V_{(V)b} q^{ab} - \int_{\Sigma} \epsilon_q V_{(L)} D_a V_{(V)b} q^{ab}, \quad (\text{A3})$$

where s_a is the volume element of the $(n-1)$ -dimensional boundary $\partial\Sigma$ of Σ . Since the second term of Eq. (A3) vanishes due to the condition $D^a V_{(V)a} = 0$, the inner product (V, W) vanishes if $V_{(L)}$ and $V_{(V)b}$ satisfy some appropriate boundary conditions at the boundary $\partial\Sigma$ of Σ so that the first term of Eq. (A3) vanishes. In this sense, the scalar part (the first term in Eq. (A1)) and the vector part (the second term in Eq. (A1)) orthogonal to each other. Geometrically, the decomposition of 1-forms, and more generally p -forms, leads via de Rham’s theorem to a characterization of topological invariants of Σ (i.e., Betti Numbers) [28].

In this appendix, it is assumed that the n -dimensional space Σ is *closed* (compact manifolds without boundary) following York’s discussions. Although the decomposition discussed here

will also be valid for the other n -dimensional spaces Σ with the boundary $\partial\Sigma$ with some appropriate boundary conditions at $\partial\Sigma$, in this Appendix, we choose the closed spaces as the topology of Σ for mathematical convenience. Through this assumption, in this Appendix, we consider the TT-decomposition (transverse traceless decomposition) of a symmetric tensor ψ^{ab} on Σ , which is defined by

$$\psi^{ab} = \psi_{TT}^{ab} + \psi_L^{ab} + \psi_{Tr}^{ab}, \quad (\text{A4})$$

where the longitudinal part is

$$\psi_L^{ab} := D^a W^b + D^b W^a - \frac{2}{n} q^{ab} D_c W^c =: (LW)^{ab} \quad (\text{A5})$$

and the trace part is

$$\psi_{Tr}^{ab} := \frac{1}{n} \psi q^{ab}, \quad \psi := q_{cd} \psi^{cd}. \quad (\text{A6})$$

Let us suppose that both an arbitrary symmetric tensor field ψ^{ab} and the metric q_{ab} are C^∞ tensor fields on Σ . First, we define ψ_{TT}^{ab} in accordance with Eq. (A4) by

$$\psi_{TT}^{ab} := \psi^{ab} - \frac{1}{n} \psi q^{ab} - (LW)^{ab}. \quad (\text{A7})$$

We note that the tensor ψ_{TT}^{ab} is traceless, i.e.,

$$q_{ab} \psi_{TT}^{ab} = 0 \quad (\text{A8})$$

by its construction (A7). Further, we require the transversality on the tensor field ψ_{TT}^{ab} , i.e.,

$$D_b \psi_{TT}^{ab} = 0. \quad (\text{A9})$$

Equation (A9) leads to a covariant equation of the vector field W^a in Eq. (A7) as

$$D_a (LW)^{ab} = D_a \left(\psi^{ab} - \frac{1}{n} \psi q^{ab} \right). \quad (\text{A10})$$

The explicit expression of (A10) is given by

$$\mathcal{D}^{bc} W_c = D_a \left(\psi^{ab} - \frac{1}{n} \psi q^{ab} \right), \quad (\text{A11})$$

where the derivative operator \mathcal{D}^{bc} is defined by

$$\mathcal{D}^{bc} := q^{bc} \Delta + \left(1 - \frac{2}{n} \right) D^b D^c + R^{bc}, \quad \Delta := D^a D_a, \quad (\text{A12})$$

where R^{bc} is the Ricci curvature on (Σ, q_{ab}) .

The basic properties of Eq. (A11) are also discussed by York [26]. The operator \mathcal{D}^{ab} defined by Eq. (A12) is linear and second order by its definition. Further, this operator is strongly elliptic, negative-definite, self-adjoint, and its “harmonic” functions are always orthogonal to the source (right-hand side) in Eq. (A11). Here, “harmonic” functions of \mathcal{D}^{ab} means functions which belong the kernel of the operator \mathcal{D}^{ab} . Moreover, he showed that Eq. (A11) will always possess solutions W^a which is unique up to conformal Killing vectors. Due to this situation, in this paper, we assume that the Green function $(\mathcal{D}^{-1})_{ab}$ defined by

$$(\mathcal{D}^{-1})_{ab} \mathcal{D}^{bc} = \mathcal{D}^{bc} (\mathcal{D}^{-1})_{ab} = \delta_a^c \quad (\text{A13})$$

exists through appropriate boundary conditions at the boundary $\partial\Sigma$ of Σ . Although York’s discussions are for the case of the *closed* space Σ , we review his discussions here. In this

review, we explicitly write the boundary terms which are neglected by the closed boundary condition to keep the extendibility to non-closed Σ case of discussions in our mind.

The ellipticity of an operator depends only upon its *principal part*, i.e., the highest derivatives acting on the unknown quantities which it contains. To see the ellipticity of an operator, we consider the replacement of the each derivative operator D_a occurring in its principal part by an arbitrary vector V_a . Through this replacement, the principal part of the operator defines a linear transformation σ_v . The operator is said to be elliptic if σ_v is an isomorphism [29]. In the present case,

$$[\sigma_v(\mathcal{D})]^{ab} = V^b V^a + q^{ab} V_c V^c. \quad (\text{A14})$$

Here, σ_v operates on vector X_a and defines a vector-space isomorphism when the determinant of σ_v is non-vanishing for all non-vanishing V^a . We can verify $\det \sigma_v \neq 0$, for example, by choosing $V^a = (\partial/\partial x^\mu)^a$ in a local Cartesian frame $\{x^\mu\}$. The operator is said to be *strongly elliptic* if all the eigenvalues of σ_v are nonvanishing and have the same sign. This is easily checked and \mathcal{D}^{ab} is strongly elliptic.

To show that \mathcal{D}^{ab} is negative definite, we consider the inner product (A2) of the vector field $\mathcal{D}W^a := \mathcal{D}^{ab}W_b$ and W^a :

$$\begin{aligned} (W, \mathcal{D}W) &= \int_{\Sigma} \epsilon_q q_{ab} W^a \mathcal{D}^{bc} W_c \\ &= \int_{\partial\Sigma} s_c W_b (LW)^{bc} - \frac{1}{2} \int_{\Sigma} \epsilon_q (LW)_{bc} (LW)^{bc}, \end{aligned} \quad (\text{A15})$$

where we use the fact that the tensor $(LW)^{bc}$ is symmetric and traceless. Eq. (A15) shows that the operator \mathcal{D}^{ab} has the negative eigenvalues in the case where the first term (boundary term) in Eq. (A15) is neglected, unless $(LW)^{bc} = 0$. The self-adjointness of the operator \mathcal{D}^{ab} is follows from a similar argument in which one integrates by parts twice:

$$\begin{aligned} \int_{\Sigma} \epsilon_q q_{ab} V^a (\mathcal{D}W)^b &= \int_{\partial\Sigma} s_c \left[V_b (LW)^{bc} - (LV)^{bc} W_b \right] \\ &\quad + \int_{\Sigma} \epsilon_q W_b \mathcal{D}^{bc} V_c \end{aligned} \quad (\text{A16})$$

for any vectors V and W , where we use the fact that the tensor $(LW)^{ab}$ and $(LV)^{ab}$ are symmetric and traceless. Eq. (A16) shows that the operator \mathcal{D}^{ab} is self-adjoint if the first term (boundary term) in Eq. (A16) is negligible.

When we can neglect the boundary terms in Eq. (A15), the right-hand side of (A15) can vanish only if $(LW)^{ab} = 0$. This means either $W^a = 0$ or W^a is a conformal Killing vector (or Killing vector) of the metric q_{ab} . The condition for a conformal Killing vector is, of course, not satisfied for an arbitrary metric but this is given by

$$\mathcal{L}_W q_{ab} = \lambda q_{ab} \quad (\text{A17})$$

for some scalar function λ , where \mathcal{L}_W denotes the Lie derivative with respect to W^a . Taking the trace of both sides, we find

$$\lambda = \frac{2}{n} D_c W^c. \quad (\text{A18})$$

Therefore, W^a is a conformal Killing vector if and only if

$$D^a W^b + D^b W^a - \frac{2}{n} q^{ab} D_c W^c \equiv (LW)^{ab} = 0. \quad (\text{A19})$$

It follows that the only nontrivial solutions of $\mathcal{D}^{ab}W_b = 0$ are conformal Killing vectors if they exist. Hence the nontrivial “harmonic” functions of \mathcal{D}^{ab} are conformal Killing vectors.

We shall now show that even if these “harmonic” solutions exist, they are always orthogonal to the right-hand side of (A10) and, hence, can cause no difficulties in solving equation (A10) by an eigen function expansion. Denote the conformal Killing vectors by $W^a = C^a$, where by definition $(LC)^{ab} = 0$. Form the scalar product of the right-hand side of (A10) with C and integrate by parts to find

$$\begin{aligned} & \int_{\Sigma} \epsilon_q q_{ac} D_b \left(\psi^{ab} - \frac{1}{n} q^{ab} \psi \right) C^c \\ &= \int_{\partial\Sigma} s_b \left(\psi^{ab} - \frac{1}{n} q^{ab} \psi \right) C_a - \frac{1}{2} \int_{\Sigma} \epsilon_q \left(\psi^{ab} - \frac{1}{n} q^{ab} \psi \right) (LC)_{ab} \\ &= 0, \end{aligned} \tag{A20}$$

where we use the fact that $\psi^{ab} - \frac{1}{n} q^{ab} \psi$ is symmetric and traceless and we also neglect the boundary term. Hence the source in Eq. (A11) is in the domain of $(\mathcal{D}^{-1})^{ab}$ and $(\mathcal{D}^{-1})^{ab}$ gives the solution to Eq. (A11) even in the presence of conformal symmetries.

These results also show that the solution to Eq. (A11) must be unique up to conformal Killing vector fields. Since only $(LW)^{ab}$ enters in the definition (A7) of ψ_{TT}^{ab} , conformal Killing vectors cannot affect ψ_{TT}^{ab} .

The orthogonality of ψ_{TT}^{ab} , $(LW)^{ab}$, and $\frac{1}{n} \psi q^{ab}$ is easily demonstrated. We see readily that $\frac{1}{n} \psi q^{ab}$ is pointwise orthogonal to $(LW)^{ab}$ and to ψ_{TT}^{ab} , as $(LW)^{ab}$ and ψ_{TT}^{ab} are both trace-free. To show that ψ_{TT}^{ab} and $(LV)^{ab}$ are orthogonal for any vector V^a and any TT tensor, we only show that

$$\begin{aligned} \int_{\Sigma} \epsilon_q q_{ac} q_{bd} (LW)^{ab} \psi_{TT}^{cd} &= \int_{\partial\Sigma} s_a \left(2W_b \psi_{TT}^{ab} \right) - \int_{\Sigma} \epsilon_q \left(2W_b D_a \psi_{TT}^{ab} \right) \\ &= 0, \end{aligned} \tag{A21}$$

where we use the fact that the tensor ψ_{TT}^{ab} is symmetric, traceless, and transverse (A9). We also neglect the boundary term in Eq. (A21). Thus, we conclude that the decomposition defined by (A7) exists, is unique, and is orthogonal.

One can further decompose the vector W^a uniquely into its transverse and longitudinal parts with respect to the metric q_{ab} . This splitting is orthogonal, as in Eq. (A1).

Since the above discussions are for the case of a closed spaces Σ , careful discussions on the boundary terms, which are neglected in the case of a closed Σ , is necessary if we extend the above arguments to the case of a non-closed Σ . However, we do not go into these detailed issues. Instead, in the main text, we assume that the existence of the Green function of the derivative operator \mathcal{D}^{ab} and use the transverse-traceless decomposition for an arbitrary symmetric tensor on Σ discussed here.